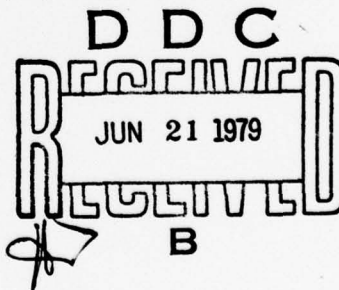(12) LEVEL II

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COLLOCATION APPROXIMATION TO EIGENVALUES
OF AN ORDINARY DIFFERENTIAL EQUATION:
THE PRINCIPLE OF THE THING

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ABSTRACT

It is shown that simple eigenvalues of an m-th order
ordinary differential equation are approximated within
$O(|\Delta|^{2k})$ by collocation at Gauss points with piecewise
polynomial functions of degree $< m+k$ on a mesh Δ. The
same rate is achieved by certain averages in case the eigen-
value is not simple. The argument relies on an extension
and simplification of Osborn's recent results concerning
the approximation of eigenvalues of compact linear maps.

delta

absolute value of delta to the 2k power

AMS(MOS) Subject Classification 65L15, 65J05

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Significance and Explanation

Collocation at Gauss points has become a popular method for the solution of boundary value problems for ordinary differential equations, both linear and nonlinear. It is therefore natural to consider the method for the solution of the corresponding eigenvalue problem. But, theory and numerical experiments reported in the literature left open the question of just how well eigenvalues would be approximated by this method. That is, proven convergence rates were much lower than one would hope for or could demonstrate in careful numerical experiments. The present report settles these questions by proving optimal convergence rates for the method. The argument required a reworking and strengthening of the existing theory for the approximation of eigenvalues of compact linear maps.

COLLOCATION APPROXIMATION TO EIGENVALUES
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0. Introduction. The eigenvalue problem we consider is of the form

(1) $(Mx)(t) = \lambda(Nx)(t)$ for $t \in [0,1]$, $\beta_i x = 0$, $i = 1,\ldots,m$,

where $(\lambda, x) \in \mathbb{C} \times C^{(m)}[0,1]$ is being sought and

(2) $(Mx)(t) := (D^m x)(t) + \sum_{i<m} a_i(t)(D^i x)(t)$

$(Nx)(t) := \sum_{i<m} b_i(t)(D^i x)(t)$

with $(\beta_i)_1^m$ a sequence of linear functionals on $C^{(m-1)}[0,1]$.

We approximate this problem by collocation, as follows. We choose a (strict) partition $\Delta := (t_r)_0^\ell$ of $[0,1]$,

$0 = t_0 < \ldots < t_\ell = 1$,

and, based on it, a sequence $(\tau_i)_1^{k\ell}$ of collocation points, k to each interval $[t_r, t_{r+1}]$,

(3) $\tau_{kr+i} := (t_r + t_{r+1} + \rho_i \Delta t_r)/2$, $i = 1,\ldots,k$,

with $(\rho_i)_1^k$ a fixed sequence of points in the "standard" interval $[-1^+, 1^-]$. Then, we seek $\lambda_\Delta \in \mathbb{C}$ and $x_\Delta \in \mathbb{P}_{k+m,\Delta}^m := \mathbb{P}_{k+m,\Delta} \cap C^{(m-1)}[0,1]$ satisfying

(1_Δ) $(Mx_\Delta)(\tau_i) = \lambda_\Delta(Nx_\Delta)(\tau_i)$, $i = 1,\ldots,k\ell$, $\beta_i x_\Delta = 0$, $i = 1,\ldots,m$.

Here, $\mathbb{P}_{r,\Delta}$ consists of all functions on $[0,1]$ which, on each interval $[t_j, t_{j+1}]$, coincide with some polynomial of order r, or, of degree $< r$.

In this paper, we bound the difference between the eigenelement (λ, x) and its collocation approximation $(\lambda_\Delta, x_\Delta)$, as a function of the maximum mesh $|\Delta| := \max_i \Delta t_i$, the ascent of λ, and the choice of the collocation pattern $(\rho_i)_i^k$. In particular, we show, in Theorem 3.1, that

(4) $\lambda_\Delta = \lambda + \mathcal{O}(|\Delta|^{2k})$

in case λ has ascent 1 and $(\rho_i)_1^k$ are the Gauss-Legendre points. We also show that it is not possible to obtain this result as an application of Osborn's [11] general theory. Further, we show that (4) holds also when λ has ascent greater than 1 provided λ_Δ is replaced by an average of certain nearby approximate eigenvalues. We also obtain $\mathcal{O}(|\Delta|^{2k})$ approximations to breakpoint values of the eigenvector x.

Here is an outline of the paper. In Section 1, we identify collocation as a particular projection method applied to the eigenvalue problem

$$\lambda T y = y$$

for a certain compact linear map T on an appropriate Banach space. In Section 2, we apply Osborn's nice results to this projection method, indicate that one cannot obtain (4) in that way, but defer a proof of this claim to Section 4. We then modify Osborn's analysis appropriately in order to relate, in Theorem 2.2, the error $\lambda - \lambda_\Delta$ to numbers of the form

$$\langle (T-T_\Delta)\varphi_\Delta, \varphi^* \rangle,$$

with φ_Δ an approximate (generalized) eigenvector and φ^* a (generalized) eigenvector of the adjoint problem. In Section 3, these numbers are shown to be $O(|\Delta|^{2k})$ for collocation at Gauss points, which leads to (4). Section 3 also contains a short discussion of related results.

We have deferred discussion of numerical examples to a companion paper [3]. We found only one real example of an ordinary differential equation with an eigenvalue of ascent greater than one in the literature. Yet, existing theory (and the theory developed in the present paper) give convergence results which strongly depend on the ascent. The three examples we give do show that the proven convergence rates cannot be improved. We also offer an explanation of sorts for the curious way in which approximate eigenvalues have, in some examples, been observed to converge to an eigenvalue of ascent greater than one.

In the process of writing the present paper, we have reexamined the at times convoluted arguments in de Boor and Swartz [2] for the superconvergence of Gauss point collocation approximations at knots. This has led us to a whole family of projectors onto $\mathbb{P}^m_{m+k,\Delta}$, of which interpolation at the Gauss collocation points is only one example, which all lead to $O(|\Delta|^{2k})$ approximations at knots and therefore give $O(|\Delta|^{2k})$ approximations to simple eigenvalues. We discuss these ideas in [4], where we also give an alternative proof of Theorem 3.1 which makes no reference to the arguments in [2].

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1. Translation to an abstract setup. We would like to discuss collocation in the setting of Osborn [11], Vainikko [12] et al., i.e., as a problem of approximating the eigenelements of a compact map T on some Banach space Y, and therefore assume that 0 is not an eigenvalue of (0.1) and that the coefficients of M are continuous (later, we will assume them to be quite smooth). Then M^{-1} exists as an integral operator on $Y := \mathbb{L}_2[0,1]$,

(1) $\qquad (M^{-1}y)(t) = \int_0^1 G(t,s) \, y(s) \, ds$,

with G Green's function for the problem

$\qquad\qquad (Mx)(t) = y(t)$ for $t \in [0,1]$, $\beta_i x = 0$, $i = 1,\ldots,m$.

Further, (λ,x) is an eigenelement of (0.1) if and only if $(1/\lambda, Mx)$ is an eigen-element of the compact linear map

(2) $\qquad\qquad T := N\,M^{-1}$

on $Y = \mathbb{L}_2[0,1]$. Note that T is given as an integral operator with a piecewise continuous kernel $\mathcal{T}$,

(3) $\qquad\qquad \mathcal{T}(t,s) := \sum_{i=0}^{m-1} b_i(t) \, (\partial/\partial t)^i G(t,s)$,

therefore compact as a map from $\mathbb{L}_2$ into C. In fact, T maps $\mathbb{L}_2$-bounded sets to uniformly Lipschitz continuous sets.

Such a change transforms (0.1_Δ) into the problem of finding $(\lambda_\Delta, y_\Delta) \in \mathbb{C} \times V_\Delta$ such that

$\qquad\qquad y_\Delta(\tau_i) = \lambda_\Delta (Ty_\Delta)(\tau_i)$, $\qquad i = 1,\ldots,k\ell$,

with

(4) $\qquad\qquad V_\Delta := \{Mz : z \in \mathbb{P}_{m+k,\Delta}^m$, $\beta_i z = 0$, $i = 1,\ldots,m\}$.

We convert this into an ordinary eigenvalue problem involving some operator T_Δ on Y by introducing the linear projector P_Δ which carries $C_\Delta[0,1] (:= C[t_0,t_1] \times \ldots \times C[t_{\ell-1}, t_\ell])$ into V_Δ by associating with each $y \in C_\Delta[0,1]$ the unique element $P_\Delta y \in V_\Delta$ for which

$\qquad\qquad (P_\Delta y)(\tau_i) = y(\tau_i)$, $\quad i = 1,\ldots,k\ell$.

Then the two conditions: $f \in V_\Delta$ and $f(\tau_i) = g(\tau_i)$, $i = 1,\ldots,k\ell$, are equivalent to the one condition: $f = P_\Delta g$. The approximate eigenvalue problem (0.1_Δ) is therefore equivalent to the problem of finding $(\lambda_\Delta, y_\Delta) \in \mathbb{C} \times Y$ such that

(5) $y_\Delta = \lambda_\Delta T_\Delta y_\Delta$

with

(6) $T_\Delta := P_\Delta T$.

Of course, it is not clear a priori that P_Δ is even defined. But note that, by

definition, $P_\Delta y$ is the unique element in V_Δ, i.e., of the form Mx_Δ with $x_\Delta \in \mathbb{P}^m_{k+m,\Delta}$

and $\beta_i x_\Delta = 0$, $i = 1,\ldots,m$, which agrees with y at the τ_i's. According to

Theorem 3.1 of de Boor and Swartz [2], there exists a positive $\underline{const}_M$ depending on M

(and ρ) so that, for all partitions Δ with

$$|\Delta| := \max_i \Delta t_i \le \underline{const}_M$$

and all $y \in C_\Delta[0,1]$, the collocation equation

$$(Mx_\Delta)(\tau_i) = y(\tau_i), \quad i = 1,\ldots,k\ell, \quad \beta_i x_\Delta = 0, \quad i = 1,\ldots,m,$$

has exactly one solution, x_Δ , in $\mathbb{P}^m_{k+m,\Delta}$. (To be precise, de Boor and Swartz [2]

assume additionally that $(\beta_i)^m_1$ is linearly independent over $\mathbb{P}_m$, but this assumption

was shown to be superfluous by Wittenbrink [16]. Also, both papers only consider real

valued functions, but the extension to complex valued functions is trivial. Wittenbrink's

formulation would also appear to be appropriate for problems in which the β_i also depend

on λ.) This insures that P_Δ is defined for all Δ with $|\Delta|$ sufficiently small,

and allows us to conclude from [2] that P_Δ converges to the identity pointwise on $C[0,1]$.

In fact, if $(\beta_i)^m_1$ is linearly independent over $\mathbb{P}_m = \ker D^m$, then $P_\Delta = (D^m M^{-1})^{-1} P(D^m M^{-1})$, with P the linear projector taking $D^m x$ to $D^m x_\Delta$ and shown in [2]

to be bounded on $C[0,1]$ independently of Δ ; hence P_Δ is then bounded on $C[0,1]$ inde-

pendently of Δ . In the more general case, an argument like Wittenbrink's (see [16;

proof of Theorem 2]) shows that

(7) $\| (1 - P_\Delta)y \|_\infty \le \underline{const} \, \| (1 - Q_\Delta)D^m M^{-1} y \|_\infty$

for some $\underline{const}$ independent of Δ and with Q_Δ interpolation from $\mathbb{P}_{k,\Delta}$ at the colloca-

tion points $(\tau_i)^{k\ell}_1$.

In conclusion, not only is P_Δ defined for all small $|\Delta|$, but, since T maps $\mathbb{L}_2$

compactly into C , we have $T_\Delta = P_\Delta T$ defined and uniformly convergent to T as $|\Delta| \to 0$.

2. A general result.   In this section, we recall Osborn's results and then
rephrase his arguments to obtain a simple yet useful formulation of the error in eigen-
value approximations by projection (and other) methods. This formulation makes it evident
(as Osborn's or Vainikko's does not) why eigenvalue approximations by collocation are of
such high order.

We recall from Osborn [11] (or from a standard reference such as Kato [8]) that,
associated with each nonzero eigenvalue μ of a compact linear map T on the Banach
space Y is the invariant subspace

(1) $S := \ker{(\mu - T)}^{\alpha}$,

i.e., the kernel or nullspace of the linear map $(\mu - T)^{\alpha}$, where α , the ascent or rank
of μ , is the smallest integer for which $\ker(\mu - T)^{\alpha} = \ker(\mu - T)^{\alpha+1}$. S is finite
dimensional, say

            $a := \dim S =:$ algebraic multiplicity of μ ,

while the geometric multiplicity of μ is the number $g := \dim \ker(\mu - T)$. The elements
of $\ker(\mu - T)\backslash\{0\}$ are the eigenvectors of T , while those of $S\backslash\ker(\mu - T)$ are called
root vectors or generalized eigenvectors. S is the range of the linear projector E
given, e.g., by the formula

(2) $E := \dfrac{1}{2\pi i} \displaystyle\int_{\Gamma} (z - T)^{-1}dz$,

with Γ any circle in the complement of the spectrum of T and enclosing in its interior
μ , but no other eigenvalue of T . The linear projector E can also be written in the
form

(3) $Ey = \displaystyle\sum_{i=1}^{a} \langle y, \varphi_i^* \rangle \varphi_i$,

with $(\varphi_i)_1^a$ any basis for S and $(\varphi_i^*)_1^a$ the corresponding dual basis in
            $S^* := \ker(\mu - T^*)^{\alpha}$,

i.e., $\varphi_i^* \in S^*$, all i , and $\langle \varphi_i, \varphi_j^* \rangle = \delta_{ij}$, all i,j. Here, T^* is the (Banach)-
adjoint or dual of T , i.e., the linear map on the topological dual Y^* of Y which is
characterized by

            $\langle Ty, y^* \rangle = \langle y, T^* y^* \rangle$, all $y \in Y$, $y^* \in Y^*$,

and $\langle \, , \, \rangle$ denotes the natural pairing between Y and Y^* . In particular, S^* is the
invariant subspace of T^* corresponding to μ and has also dimension a .

In terms of this notation, Osborn [11] establishes (among others) the following facts.

Theorem 2.1 (Osborn) Let (T_ν) be a sequence of compact linear maps on the Banach space Y converging uniformly to the linear map T, and let μ be a (nonzero) eigenvalue of T. Then, for all large ν,

$$(4) \qquad E_\nu := \frac{1}{2\pi i} \int_\Gamma (z - T_\nu)^{-1} dz$$

is defined; its range, S_ν, has dimension a and satisfies

$$(5) \qquad gap(S, S_\nu) \le \text{const} \, \| (T - T_\nu) \big|_S \|$$

with

$$(6) \qquad gap(U,V) := \max \left\{ \sup_{x \in U} \frac{dist(x,V)}{\|x\|} \;,\; \sup_{x \in V} \frac{dist(x,U)}{\|x\|} \right\}.$$

Further, if $\hat\mu$ is one of the eigenvalues of T_ν enclosed by Γ, then

$$(7) \qquad |\mu - \hat\mu|^a \le \text{const} \, \Big(\| (T - T_\nu) \big|_S \| \| (T^* - T_\nu^*) \big|_{S^*} \| + \sum_{i,j=1}^{a} |\langle (T - T_\nu) \varphi_i, \varphi_j^* \rangle | \Big).$$

However, the average

$$\bar\mu := (\mu_1 + \ldots + \mu_a)/a \,,$$

comprised of the a eigenvalues of T_ν (counting algebraic multiplicities) enclosed by Γ, admits the closer estimate

$$(8) \qquad |\mu - \bar\mu| \le \underline{\text{const}} \, \Big\{ \| (T - T_\nu) \big|_S \| \| (T^* - T_\nu^*) \big|_{S^*} \| + \sum_{i=1}^{a} |\langle (T - T_\nu) \varphi_i, \varphi_i^* \rangle | \Big\}.$$

This is a powerful theorem, and Osborn makes good use of it in a variety of applications described at the end of his paper, mostly to Galerkin approximations. But the theorem does not suffice to explain the high accuracy of eigenvalue approximations obtained by collocation. The chief difficulty lies in the fact that, in collocation (as translated, in the preceding section, into this abstract setup), T_ν^* does not approximate T^* sufficiently well, see Section 4.

We obtain more direct control by comparing matrix representations for $T\big|_S$ and $T_\nu \big|_{S_\nu}$, as is done, in effect, in Osborn [11] to get the estimates (7) and (8), and is done, quite explicitly, in Atkinson [1] and in Kreiss [9]; but we do it a little bit differently.

Let J be the matrix representation of $T\big|_S$ with respect to the basis $(\varphi_i)_1^a$ for S. Since $gap(S, S_\nu) \to 0$, $E\big|_{S_\nu}$ is 1-1 for all large ν, and then there is a corresponding basis $(\varphi_{i,\nu})_1^a$ for S_ν with

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(9) $E\varphi_{i,\nu} = \varphi_i$, $i = 1,\ldots,a$.

Since, by (3), $\varphi_j^* = \varphi_j^* E$, we see that

$$\langle \varphi_{i,\nu}, \varphi_j^* \rangle = \langle E\varphi_{i,\nu}, \varphi_j^* \rangle = \langle \varphi_i, \varphi_j^* \rangle = \delta_{ij}$$

hence $y = \Sigma_i \langle y, \varphi_i^* \rangle \varphi_{i,\nu}$ for all $y \in S_\nu$, thus the matrix representation J_ν for $T_\nu\big|_{S_\nu}$ with respect to $(\varphi_{i,\nu})_1^a$ has (i,j)-entry

$$(J_\nu)_{ij} = \langle T_\nu \varphi_{j,\nu}, \varphi_i^* \rangle , \quad i,j = 1,\ldots,a .$$

But the entries of J are given by a similar expression. We have

$$(J)_{ij} = \langle T\varphi_j, \varphi_i^* \rangle = \langle TE\varphi_{j,\nu}, \varphi_i^* \rangle = \langle ET\varphi_{j,\nu}, \varphi_i^* \rangle = \langle T\varphi_{j,\nu}, \varphi_i^* \rangle ,$$

$$i,j = 1,\ldots,a ,$$

using the fact that T commutes with its spectral projector E . Consequently,

(10) $\| J - J_\nu \| \leq \underline{const}_{\|\cdot\|} \max_{i,j} | \langle (T-T_\nu)\varphi_{j,\nu}, \varphi_i^* \rangle |$.

    We gather these facts into a theorem .

    Theorem 2.2.   With the assumptions and notations of Theorem 2.1, let J be the matrix representation of $T\big|_S$ with respect to some basis $(\varphi_i)_1^a$ of S . Then, $T_\nu\big|_{S_\nu}$ is similar to a matrix J_ν close to J in the sense that

(11) $(J - J_\nu)_{ij} = \langle (T - T_\nu)\varphi_{j,\nu}, \varphi_i^* \rangle$, $i,j = 1,\ldots,a$,

with $(\varphi_{j,\nu})_1^a$ the basis for S_ν for which $E\varphi_{j,\nu} = \varphi_j$, all j , and $(\varphi_i^*)_1^a$ the basis for S^* dual to $(\varphi_i)_1^a$ (and to $(\varphi_{i,\nu})_1^a$).

    It follows that the eigenvalues of T_ν close to μ are those of the matrix J_ν (including algebraic multiplicity and structure) and, since J has μ as an eigenvalue in just the same way as T does (including algebraic multiplicity and structure), the approximation properties of the process can be read off from standard perturbation arguments which compare the eigenvalues of a matrix J with those of a perturbation J_ν , e.g., from Wilkinson [13], in the manner practiced by Atkinson [1] and Kreiss [9] .

    In particular,

(12) $|\mu - \mu_i| \leq const \| J - J_\nu \|^{1/\alpha}$

for any of the a   eigenvalues μ_i of J_ν , while

(13)
$$\left|\mu - (\text{trace } J_\nu)/a\right| = \left|\text{trace}(J - J_\nu)\right|/a \leq \underline{\text{const }} \|J - J_\nu\|$$

and also

(14)
$$\left|1/\mu - (\text{trace } J_\nu^{-1})/a\right| \leq \underline{\text{const }} \|J - J_\nu\| \ .$$

In fact, with $\mu_1,\ldots,\mu_a$ the eigenvalues of J_ν counting algebraic multiplicities, and $\sigma_0,\ldots,\sigma_a$ the elementary symmetric functions in a variables, we see, by comparing coefficients of the characteristic polynomials of J and J_ν, that

(15)
$$\sigma_j(\mu_1,\ldots,\mu_a) = \sigma_j(\mu,\ldots,\mu) + O(\|J - J_\nu\|), \quad \text{all } j \ .$$

Finally, with g the geometric multiplicity of μ, there exists an eigenvalue μ_i of J_ν (and T_ν) so that

(16)
$$\mu = \mu_i + O(\|J - J_\nu\|^{g/a}) \ .$$

This is a better rate than (12) except when all Jordan blocks of J have the same size.

The estimate (16) appears in Wilkinson [13, p. 81], while (13), (14) and (15) are obvious. The proof of (12) (via Gershgorin's circle theorem) is left to the reader in Wilkinson [13, p. 80-81]. Atkinson [1] gives, so he says, a shorter proof. We record our own version, for the record.

Since $(\mu - J)^\alpha = 0$,

$$\left|\mu - \mu_i\right|^\alpha \leq \|(\mu - J_\nu)^\alpha\| = \|(\mu - J_\nu)^\alpha - (\mu - J)^\alpha\| \leq \underline{\text{const}}_{\|J\|, \|J_\nu\|} \|J - J_\nu\|$$

using the fact that the map $A \mapsto A^\alpha$ is locally Lipschitz continuous. (Indeed, $A^\alpha - B^\alpha = \sum_{j=0}^{\alpha-1} A^{\alpha-j-1}(A-B)B^j$, hence $\|A^\alpha - B^\alpha\| \leq [\sum_{j=0}^{\alpha-1} \|A\|^{\alpha-j-1}\|B\|^j]\|A-B\|$.) This proves (12), given that $J_\nu \to J$.

Note that we recover (7)-(8) of Osborn's theorem directly from (12)-(13) because of (10) and since

(17)
$$(J - J_\nu)_{ij} = \langle (T - T_\nu)(E|_{S_\nu})^{-1}\varphi_j, \varphi_i^* \rangle$$
$$= \langle (E|_{S_\nu})^{-1}\varphi_j - \varphi_j, (T-T_\nu)^*\varphi_i^* \rangle + \langle (T-T_\nu)\varphi_j, \varphi_i^* \rangle,$$

while

(18)
$$\|(E|_{S_\nu})^{-1}\varphi - \varphi\| \leq \|1 - E\|\text{dist}((E|_{S_\nu})^{-1}\varphi, \text{ ran } E)$$
$$\leq \|1 - E\|\underline{\text{const }}\|(T-T_\nu)|_S\|\|\varphi\|,$$

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the last by (5) and because $\| (E|_{S_\nu})^{-1}\varphi \| \leq \underline{const}\|\varphi\|$ for all sufficiently large ν. Here we have used the facts that $(E|_{S_\nu})^{-1}\varphi - \varphi = (1-E)(E|_{S_\nu})^{-1}\varphi$, and that E is a linear projector onto S, hence $\|(1-E)v\| \leq \|1-E\|\text{dist}(v,S)$ while, for $v \in S_\nu$, $\text{dist}(v,S) \leq \text{gap}(S,S_\nu)\|v\|$.

It remains, for any particular approximation sequence (T_ν), to estimate $\|J - J_\nu\|$, i.e., to estimate expressions of the form

(19) $$\langle (T-T_\nu)(E|_{S_\nu})^{-1}\varphi, \varphi^* \rangle \ ,$$

with $\varphi \in S$, $\varphi^* \in S^*$. The estimate

(20) $$\|J - J_\nu\| \leq \underline{const}\|(T - T_\nu)|_S\|$$

is, of course, immediate from (17) and (18) and provides, in conjunction with (12)-(16), first assurance that the eigenvalues of T are, indeed, approximated well by those of T_ν. It also makes the important point that the error in the eigenvalue approximation achieved by a particular discretization method is at least of the same order as the error in the corresponding discrete solution of the corresponding nonsingular equation. But, this estimate ignores the fact that the linear functional φ^* is being applied to a residual error, i.e., to an element of the form $(T - T_\nu)y$, a fact that, at times, leads to sharper estimates. For example, assume that T_ν is obtained from T by projection, i.e., $T_\nu = P_\nu T$ for some projector P_ν. Then, as in (17) and (18),

(21)
$$\left| \langle (T-T_\nu)(E|_{S_\nu})^{-1}\varphi, \varphi^* \rangle \right|$$
$$= \left| \langle (E|_{S_\nu})^{-1}\varphi - \varphi, \varphi^*(1-P_\nu)T \rangle + \langle (1-P_\nu)T\varphi, \varphi^*(1-P_\nu) \rangle \right|$$
$$\leq \underline{const} \ \|(1-P_\nu)|_S\| \|(1-P_\nu^*)|_{S^*}\| \|\varphi\| \|\varphi^*\|$$

(using the facts that $(1-P_\nu)^2 = (1-P_\nu)$ and that $TS \subseteq S$). This provides an indication of the potential "double accuracy" in the eigenvalue approximation by such methods, as first proved by Vainikko [12], and also established in this generality by Osborn [11].

In the analysis of Galerkin's method, or the least-squares method, $\varphi^* \in S^*$ is given by integration,

$$\langle f, \varphi^* \rangle = \int \varphi^*(s) f(s) ds$$

for some smooth function φ^*, while $\text{ran} P_\nu^*$, i.e., the interpolation conditions for the projector $P_\nu = P_\Delta$, consists of integration against certain piecewise smooth functions.

Explicitly, then

$$\text{dist}(\varphi^*, \text{ran } P_\Delta^*) = \inf_\gamma \sup_f \left| \int \varphi^*(s)f(s)\,ds - \int \sum_i \gamma_i K_i(s)f(s)\,ds \right| / \|f\|$$

with span (K_i) akin to $\mathbb{P}_{k,\Delta}$ in approximation power. This allows the conclusion that

$$\| (1 - P_\Delta)^* |_{S^*} \| = \mathcal{O}(|\Delta|^k)$$

for those methods which, together with the more obvious fact that $\| (1 - P_\Delta) |_S \| = \mathcal{O}(|\Delta|^k)$, gives the sought-for "double accuracy" for such methods.

For collocation, though, $\text{dist}(\varphi^*, \text{ran } P_\Delta^*)$ fails to go to zero with any kind of rapidity since now

$$\text{dist}(\varphi^*, \text{ran } P_\Delta^*) = \inf_\gamma \sup_f \left| \int \varphi^*(s)f(s)\,ds - \sum_i \gamma_i f(\tau_i) \right| / \|f\| \quad ,$$

i.e., ran P_Δ^* consists of linear combinations of certain point evaluations. Therefore, the simple bound (21) only implies $\mathcal{O}(|\Delta|^k)$ convergence for Gauss point collocation. Even Osborn's more careful bound (8) only implies $\mathcal{O}(|\Delta|^{k+\hat{m}})$ convergence for Gauss point collocation, with $\hat{m}$ the difference between the order m of M and the order N (see Section 4.)

Because of this, we proved Theorem 2.2 which relates to error in the eigenvalue approximation to the value of a smooth linear functional on the residual error in a collocation approximation. For, such expressions can be shown to be $\mathcal{O}(|\Delta|^{2k})$, as is done in the next section.

3. Application to Collocation at Gauss Points.   For collocation, as described in
the introductory sections 0 and 1 , the quantities $\langle (T-T_\nu)\varphi_{j,\nu}, \varphi_i^* \rangle$ in the crucial
bound (2.10) take the form

(1) $$\int_0^1 \varphi_i^*(t)\,(T-T_\Delta)\varphi_{j,\Delta}(t)\,dt \ ,$$

and we now intend to show these to be $O(|\Delta|^{2k})$ if the collocation points are chosen
as Gauss points, i.e., if $(\rho_i)_1^k$ consists of the k zeros of the Legendre polynomial
of degree k . The analysis of collocation at Gauss points in de Boor and Swartz [2]
was based on the observation that

(2) $$\int_0^1 \varphi^*(t)\,(T-T_\Delta)\varphi(t)\,dt = O(|\Delta|^{2k})$$

in case φ^* and φ are both smooth. But, since (1) involves $\varphi_{j,\Delta}$ rather than a fixed
φ_j , we have to make more explicit just how (2) depends on φ and, because of our desire
to keep our estimates <u>mesh independent</u> , i.e., dependent only on the number $|\Delta|$, the
argument is a bit delicate.

We continue to use the notation and terms of the preceding section, but replace the
subscript "ν" by the more appropriate subscript "Δ". Also, we use the abbreviations

$$\|\varphi\|_{(r)} := \sup_{t_r < t < t_{r+1}} |\varphi(t)|$$

and

$$\|\varphi\|_{s,(r)} := \sum_{i=0}^{s} \|D^i \varphi\|_{(r)} \ .$$

Further, we assume that $(\rho_i)_i^k$ has been chosen so that

(3) $$\int_{-1}^1 p(t)\,\prod_{i=1}^k (t-\rho_i)\,dt = 0 \quad \text{for all } p \in \mathbb{P}_n \ .$$

Finally, we assume that the coefficients of M and N are smooth enough. Specifically,
we assume that

(4) $$a_i,\ b_i \in C^{(n+k)}[0,1], \quad \text{all } i \ .$$

Then, using the abbreviation

$$X^{(r)} := \{f \in C^{(r-1)}[0,1] : D^{r-1}f \text{ abs. cont.}, D^r f \in X\},$$

with X any of the spaces $\mathbb{L}_p[0,1]$, $1 \le p \le \infty$, or C[0,1], the map M^{-1} carries
$X^{(r)}$ into $X^{(r+m)}$, and N maps $X^{(r+m)}$ into $X^{(r+1)}$, for $r < n+k$. In particular,
any invariant subspace of T belonging to a nonzero eigenvalue consists of functions in

$x^{(n+k)}$, and the same holds for T^*, the adjoint of T. This insures that

(5) $$S \subseteq C^{(n+k)}, \qquad S^* \subseteq C^{(n)},$$

and implies, with (1.7), that

(6) $$\| (1 - P_\Delta) |_S \| = \mathcal{O}(|\Delta|^k) .$$

In particular, $\| J - J_\Delta \| = \mathcal{O}(|\Delta|^k)$, by (2.20), and (2.12) now gives the results of R. Winther [14] for collocation.

We also conclude from (6) and (2.18) that

(7) $$\| (E_{S_\nu})^{-1} \varphi - \varphi \|_\infty = \mathcal{O}(|\Delta|^k) \|\varphi\| \qquad \text{for } \varphi \in S .$$

Lemma 3.1. There is a constant <u>const</u> depending only on k <u>and</u> ρ so that

$$\left| \int_{t_r}^{t_{r+1}} \varphi^* (T - T_\Delta) \varphi_\Delta \right| \leq \underline{\text{const}} \; |\Delta t_r|^{n+k+1} \|\varphi^*\|_{n,(r)} \| (T - T_\Delta) \varphi_\Delta \|_{n+k,(r)} .$$

Proof. The function $(T - T_\Delta)\varphi_\Delta$ vanishes at the k collocation points $\tau_{rk+1}, \ldots, \tau_{rk+k}$ in $[t_r, t_{r+1}]$ and, by assumption (3), the polynomial $\prod_s (t - \rho_{rk+s})$ is orthogonal to $\mathbb{P}_n$ on $[t_r, t_{r+1}]$. The argument in de Boor and Swartz [2], in the proof of Theorem 4.1 (with φ^* playing the role of $G(t, \cdot)$) therefore establishes the inequality, assuming, of course, that $\varphi^* \in \mathbb{L}_\infty^{(n)}[t_r, t_{r+1}]$ and $(T - T_\Delta) \varphi_\Delta \in \mathbb{L}_\infty^{(n+k)}[t_r, t_{r+1}]$. |||

Next, consider $\| (T - T_\Delta) \varphi_\Delta \|_{n+k,(r)}$ with $\varphi_\Delta = \varphi_{j,\Delta} = (E |_{S_\Delta})^{-1} \varphi_j$. We would like to bound this number in terms of $\varphi_{j,\Delta}$ and its derivatives <u>on</u> (t_r, t_{r+1}), and this seems difficult since $T_\Delta f$ at a point depends, offhand, on f on all of $[0,1]$. But,

$$T_\Delta \varphi_{j,\Delta} = \sum_{i=1}^{a} (J_\Delta)_{ij} \varphi_{i,\Delta}$$

since $J = (\langle T_\Delta \varphi_{j,\Delta}, \varphi_i^* \rangle)$ represents $T_\Delta |_{S_\Delta}$ with respect to $(\varphi_{i,\Delta})$. Therefore, with

(8) $$x_{i,\Delta} := M^{-1} \varphi_{i,\Delta} \in \mathbb{P}_{m+k,\Delta}^m, \qquad i=1,\ldots,a,$$

we have

$$(T - T_\Delta) \varphi_{j,\Delta} = T \varphi_{j,\Delta} - \sum_i (J_\Delta)_{ij} \varphi_{i,\Delta}$$

$$= N x_{j,\Delta} - \sum_i (J_\Delta)_{ij} M x_{i,\Delta} .$$

and $J_\Delta \to J$. This shows the derivatives of $(T - T_\Delta) \varphi_{j,\Delta}$ to be expressible in terms of the derivatives of the coefficient functions of M and N (it is only here that we use the full power of our assumption (4)) and those of $x_{i,\Delta} \in \mathbb{P}_{k+m,\Delta}^m$, and, as

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$J_\Delta \to J$ (by (2.20)), it follows that

(9)
$$\|(T - T_\Delta)\varphi_{j,\Delta}\|_{n+k,(r)} \leq \underline{\text{const}}_{M,N} \max_i \|x_{i,\Delta}\|_{m+k,(r)} \ .$$

To complete the analysis, it remains to bound $\|x_{i,\Delta}\|_{m+k,(r)}$ appropriately.

<u>Lemma 3.2.</u> <u>For</u> $\varphi \in S$ <u>and</u> $x := M^{-1}\varphi$, <u>let</u> $x_\Delta := M^{-1}\varphi_\Delta$ <u>and</u> $\varphi_\Delta := (E|_{S_\Delta})^{-1}\varphi$.
<u>Then, for</u> $s=0,\ldots,m+k$,

(10)
$$\|D^s x_\Delta\|_{(r)} \leq \|D^s x\|_{(r)} + \underline{\text{const}}_x (|\Delta|/|\Delta t_r|)^k \ .$$

<u>Proof.</u> By (7) , $\|\varphi_\Delta - \varphi\|_\infty = \mathcal{O}(|\Delta|^k)$, hence

(11)
$$\|D^s(x_\Delta - x)\|_\infty = \mathcal{O}(|\Delta|^k) \qquad \text{for} \qquad s=0,\ldots,m,$$

which proves a stronger inequality than (10) for $s=0,\ldots,m$. For $s > m$, one now proceeds as in the proof of Lemma 4.1 in de Boor and Swartz [2] (with the role of x and Rx played here by x and x_Δ , respectively) to show, using (11), that

$$\|D^s(x - x_\Delta)\|_{(r)} \leq \underline{\text{const}}_x (|\Delta|/|\Delta t_r|)^k \quad \text{for} \quad s=m+1,\ldots,m+k \ ,$$

in case $x \in \mathbb{L}_\infty^{(m+n)}[t_r, t_{r+1}]$. But, by (4), $x = M^{-1}\varphi$ has even $m+k+n$ continuous derivatives. $|||$

<u>Theorem 3.1.</u> <u>Let</u> T <u>be the compact linear map on</u> $\mathbb{L}_2[0,1]$ <u>given by</u> (1.2), <u>with</u> M <u>and</u> N <u>given by</u> (0.2) <u>and satisfying</u> (4), <u>and let</u> μ <u>be a nonzero eigenvalue of</u> T <u>with corresponding invariant subspace</u> S . <u>Let</u> T_Δ <u>be the collocation approximation</u> (1.6) <u>to</u> T , <u>and assume that the collocation points</u> (ρ_i) <u>satisfy</u> (3) .

<u>For all small</u> $|\Delta|$, T_Δ <u>has an invariant subspace</u> S_Δ <u>and a matrix representation</u> J_Δ <u>of</u> $T_\Delta|_{S_\Delta}$ <u>for which</u>

$$\|J - J_\Delta\| \leq \underline{\text{const}} \ |\Delta|^{n+k} \ ,$$

<u>with</u> J <u>an appropriate matrix representation for</u> $T|_S$.

<u>Proof.</u> By Theorem 2.2, we only need to show that the integrals in (2) are $\mathcal{O}(|\Delta|^{n+k})$ under our assumptions. But, by Lemmas 3.1 and 3.2 and (5),

$$\left| \int_0^1 \varphi_i^*(T - T_\Delta)\varphi_{j,\Delta} \right| \leq \sum_r \left| \int_{t_r}^{t_{r+1}} \varphi_i^*(T - T_\Delta)\varphi_{j,\Delta} \right|$$

$$\leq \sum_r \underline{\text{const}} \ |\Delta t_r|^{n+k+1} \|\varphi_i^*\|_{n,(r)} \|(T-T_\Delta)\varphi_{\Delta,j}\|_{n+k,(r)}$$

$$\leq \sum_r \underline{\text{const}} \ |\Delta t_r|^{n+k+1} \|\varphi_i^*\|_{n,(r)} \max_j (\|x_j\|_{m+k,(r)} + (|\Delta|/\Delta t_r)^k)$$

$$\leq \underline{\text{const}} \ |\Delta|^{k+n} \ . \qquad |||$$

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For completeness, we summarize previous results concerning the approximation of eigenvalue problems (0.1) by collocation. Winther [14] was the first to publish results for the collocation of (0.1) using $\mathbb{P}^{m+1}_{k+m,\Delta}$. He proved that (the quantity he called "algebraic multiplicity" we have called "ascent")

$$|\lambda - \lambda_\Delta| = \mathcal{O}(|\Delta|^{k/\alpha}), \qquad \alpha = \text{the ascent of } \lambda ;$$

$$\|D^i(u - u_\Delta)\|_\infty = \mathcal{O}(|\Delta|^k), \quad i \leq m \qquad \text{if } \alpha = 1 .$$

In one of his numerical experiments, he collocated at the k Gauss points even though his theory no longer applied. But this experiment demonstrated that some sort of "superconvergence" might be going on for the approximate eigenvalues, although its character could not be estimated. In his thesis [15], Winther proved that, using $\mathbb{P}^m_{k+m,\Delta}$ instead,

$$|\lambda - \lambda_\Delta| = \mathcal{O}(|\Delta|^{(k+\min(n,\hat{m}))/\alpha}), \qquad \alpha = \text{ascent of } \lambda ;$$

$$\|D^i(u - u_\Delta)\|_\infty = \mathcal{O}(|\Delta|^{k+\min(n,\hat{m})}), \quad i \leq m-\hat{m} ;$$

$$\|D^i(u - u_\Delta)\|_\infty = \mathcal{O}(|\Delta|^{k+\min(n,m-i)}), \quad m-\hat{m} < i \leq m-1 ;$$

these last assuming that $\alpha = 1$. Extensive numerical work reported there also indicated that some sort of superconvergence was taking place for the approximate eigenvalues when Gauss points were used in the collocation. Lathrop [10] presented some more numerical evidence of superconvergence, for two second order operators; but still the character of the superconvergence could not be estimated. Cerutti and Parter [6] contemplated collocating the self-adjoint second order eigenvalue problem $(D^2 u)(x) = \lambda a_0(x) u(x)$ with k Gauss points using $\mathbb{P}^2_{k+2,\Delta}$. Convergence rates were not relevant to that paper; what is worth noting is their proof that the approximate eigenvalues of the non-self-adjoint matrix problem were, in fact, real.

In their monograph [7] concerned primarily with collocation using piecewise polynomials, Douglas and Dupont steered clear of straightforward collocation for the eigenvalue problem. However, they did use Bramble and Osborn's precursor [5] to Osborn's basic paper [11] to prove that for the second order problem $Mu = \lambda u$ and using $\mathbb{P}^2_{k+2,\Delta}$, one can collocate the orthogonal projection of the right-hand side onto $\mathbb{P}_{k,\Delta}$ at the Gauss points to obtain (they suppose that the ascent of λ is one)

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$$|\lambda - \lambda_\Delta| = O(|\Delta|^{2k}) ;$$

$$\|u - u_\Delta\|_0 = O(|\Delta|^{k+2}) ;$$

$$\max_j |D^i(u - u_\Delta)(x_j)| = O(|\Delta|^{2k}), \quad i = 0,1 .$$

From the start, our numerical experiments indicated quite clearly that straight-forward collocation of (0.1) at the Gauss points using $\mathbb{P}^m_{k+m,\Delta}$ should yield (at least, when the ascent of λ is one)

$$|\lambda - \lambda_\Delta| = O(|\Delta|^{2k}) ;$$

$$\max_j |D^i(u - u_\Delta)(x_j)| = O(|\Delta|^{2k}), \quad i < m ;$$

i.e., about the best one could hope for from the known results for the boundary value problem [2]. It was these experiments, of course, that prompted us to develop this paper which completes Winther's work.

4. In collocation, the adjoint problem is not approximated all that well. In this section, we elaborate on our earlier contention that Osborn's bound (2.8) is not strong enough to give the desired $O(|\Delta|^{2k})$ bound for the eigenvalue error in Gauss-point collocation. The difficulty lies with the term

$$\| (T^* - T_\Delta^*)|_{S^*} \|$$

which, so we claim, cannot be shown to be $O(|\Delta|^k)$ merely because S^* is a finite dimensional linear space of smooth function(al)s.

We begin with a simple bound on $(T^* - T_\Delta^*)|_{S^*}$ which has, in fact, nothing to do with S^* but merely relies on the obvious fact that

(1) $$\| (T^* - T_\Delta^*)|_{S^*} \| \leq \| T^* - T_\Delta^* \| = \| T - T_\Delta \| .$$

For this, let $\hat{m}$ be the difference between the order m of M and the order of N .

Lemma 4.1. Under the assumptions of Theorem 3.1,

(2) $$\| T - T_\Delta \| = O(|\Delta|^{\hat{m}}) .$$

Proof. For $f \in L_2$, we compute with (1.3) that

$$(T - T_\Delta)f = \int_0^1 (1 - P_\Delta) T(\cdot,s) f(s)\, ds.$$

Further, by (1.3) and because of the smoothness assumption (3.4), $T(\cdot,s) \in \mathbb{L}_\infty^{(m)}[0,1]$ uniformly in s , i.e.,

$$\| D^{\hat{m}} T(\cdot,s) \|_\infty \leq \underline{const}_{M,N} \quad \text{for } 0 \leq s \leq 1 .$$

Consequently, from (1.7),

$$\| (1 - P_\Delta) T(\cdot,s) \|_\infty \leq |\Delta|^{\hat{m}} \underline{const}_{M,N}$$

and (2) follows. On combining (2) and (1), we then obtain

(3) $$\| (T^* - T_\Delta^*)|_{S^*} \| = O(|\Delta|^{\hat{m}}) ,$$

which allows the conclusion from (2.8) that

(4) $$\| \mathcal{J} - \mathcal{J}_\Delta \| = O(|\Delta|^{k+\hat{m}}) ,$$

a result already in Winther [15].

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Next, we show that (3) is sharp in the sense that we cannot obtain a rate better than $O(|\Delta|^{\hat{m}})$ if all we know about S^* is that it is spanned by finitely many smooth function(al)s. For this, we exhibit a simple M, N, $(\beta_i)_0^{m-1}$, Δ and a very smooth φ^* for which

$$\|(T^* - T_\Delta^*)\varphi^*\| = \underline{\text{const}} \ |\Delta|^{\hat{m}} \ .$$

We take $\varphi^* = 1$, i.e.,

$$\varphi^* f = \int_0^1 f(t)dt, \quad \text{all} \quad f \in \mathbb{L}_2 \ .$$

Then

$$[(T^* - T_\Delta^*)\varphi^*]f = \int_0^1 \varphi^*(t) \int_0^1 (1 - P_\Delta) T(t,s) f(s) ds \ dt$$

$$= \int_0^1 [\int_0^1 (1 - P_\Delta) T(t,s)]dt \ f(s)ds \ ,$$

and, consequently,

(4) $$\|(T^* - T_\Delta^*)\varphi^*\| = \| \int_0^1 e(t,\cdot)dt \|_2$$

with

$$e(\cdot,s) := (1 - P_\Delta) T(\cdot,s) \ .$$

Next, we choose $M = D^m$, $N = D^{m-\hat{m}}$ and any appropriate $(\beta_i)_1^m$ independent over $\mathbb{P}_m$. Then, for some appropriate $p_s \in \mathbb{P}_m$,

$$T(t,s) = N_t[(t-s)_+^{m-1}/(m-1)! - p_s(t)]$$

$$= (t-s)_+^{\hat{m}-1}/(\hat{m}-1)! - q_s(t)$$

for some appropriate $q_s \in \mathbb{P}_{\hat{m}}$. Also, P_Δ is then just Q_Δ, i.e., interpolation from $\mathbb{P}_{k,\Delta}$ at the collocation points $(\tau_i)_1^{k\ell}$. Therefore, with the assumption that $k \geq m$,

$$e(\cdot,s) = (1 - Q_\Delta) T_0(\cdot,s)$$

where

$$T_0(\cdot,s) := (\cdot - s)_+^{\hat{m}-1}/(\hat{m}-1)! \ .$$

Further, $e(\cdot,s) = (1 - Q_\Delta)T_0(\cdot,s)$ vanishes on every interval (t_i, t_{i+1}) which does not contain s in its interior. Thus

$$\int_0^1 e(t,s)dt = \int_{t_i}^{t_{i+1}} e(t,s)dt$$

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with i chosen so that $[t_i, t_{i+1}]$ contains s. On this interval, we change variables, using

$$t = \sigma_i(\tau) := (t_i + t_{i+1} + \tau \, \Delta t_i)/2 \, .$$

Then we can write

$$\int_{t_i}^{t_{i+1}} e(t,s) \, dt = \int_{-1}^{1} (1 - Q) T_0(\sigma_i(\tau), s) \, \frac{\Delta t_i}{2} \, d\tau,$$

with Q polynomial interpolation to functions of τ at the points $\rho_1 < \ldots < \rho_k$, all in $(-1, 1)$. Since also, with $\sigma := \sigma_i^{-1}(s)$,

$$T_0(\sigma_i(\tau), s) = \left(\frac{\Delta t_i}{2}\right)^{\hat{m}-1} (\tau - \sigma)_+^{\hat{m}-1}/(\hat{m}-1)! \, ,$$

we therefore find that

(5)
$$\int_0^1 e(t,s) \, dt = \left(\frac{\Delta t_i}{2}\right)^{\hat{m}} /(\hat{m}-1)! \cdot F(\sigma), \quad \sigma = \sigma_i^{-1}(s), \quad t_i \le s \le t_{i+1} \, .$$

where F is the function on $[-1,1]$ given by

$$F(\sigma) := \int_{-1}^{1} (1 - Q)(\tau - \sigma)_+^{\hat{m}-1} \, d\tau \, .$$

Note that $\|F\|_2 > 0$ since, e.g., $Q(\cdot - \sigma)_+^{\hat{m}-1} = 0$ for $\sigma > \rho_k$, hence, then

$$F(\sigma) = \int_{-1}^{1} (\tau - \sigma)_+^{\hat{m}-1} \, d\tau = (1 - \sigma)^{\hat{m}}/\hat{m} > 0 \quad \text{for } \rho_k < \sigma < 1 \, .$$

Now combine (4) and (5) to get that

$$\| (T^* - T_\Delta^*)v^* \|^2 = \left(\frac{\|F\|_2}{(\hat{m}-1)!}\right)^2 \sum_{i=0}^{\ell-1} (\Delta t_i/2)^{2\hat{m}+1} \, .$$

For a uniform partition $\Delta = (i|\Delta|)_0^\ell$, therefore,

(6)
$$\| (T^* - T_\Delta^*)v^* \| = \frac{\|F\|_2}{(\hat{m}-1)! \, 2^{\hat{m}+1/2}} |\Delta|^{\hat{m}}$$

which proves our claim.

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It is shown that simple eigenvalues of an m-th order ordinary differential equation are approximated within $O(|\Delta|^{2k})$ by collocation at Gauss points with piecewise polynomial functions of degree $< m+k$ on a mesh Δ. The same rate is achieved by certain averages in case the eigenvalue is not simple. The argument relies on an extension and simplification of Osborn's recent results concerning the approximation of eigenvalues of compact linear maps.

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